About a new method for solving certain calculus of variation problems in the mathematical physics.

By Mr. Walter Ritz in Göttingen

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1 Introduction

All boundary problems of mathematical physics demand the representation of finite, continuous functions in specified finite range. A development for the power series will succeed here only in exceptional cases, and even more rarely is it numerically useful in its entire range. Eventually, even in cases where the development is essentially possible, the calculation often fails for it would require the solution of an infinite number of linear equations with an infinite number of unknowns. Polynomial developments, Fourier series etc. are much better suited for the representation of a real functiom ω(x,y,...) in a given range, because here, in terms of convergence in the entire range, only properties of continuity etc. are required, which are mostly given within boundary problems. For a numericaly given ω, the determination of the coefficients of a polynomial ω_n=a_ν+a_1x+... of a given degree n, in such a way that ω_n counts as approximation of ω, makes no problems, and the precision may be increased indefinitely at sufficiently large n. But if ω is defined as the integral of a differential equation under certain side-conditions, the calculation of the coefficients a_i may, at first, be only possible in the very specific case of a integration by rapid convergent power series. The demand rises to generally execute the approximated representation of the integral within the entire given range through a polynomial of the given degree n in this case as well, in such a way, that with growing n the precision grows indefinitly so that eventually a polynomial development of the integral results.

In that case n can be choosen small if the experience shows us, for example, that the curve we search for is only slightly curved. Fourier series etc. can also be useful; more general in most cases functions ψ_1, ψ_2, ... ψ_n, ... can be declared so that the for example through obervation known function ω of x, y, ... of the Form

$$(1.)\qquad \omega_n = \psi_0 + a_1\psi_1 + a_2\psi_2 + \cdots + a_n\psi_n$$

can be represented with sufficient precision through suitable choice of the indeterminate coefficients a_i even with n staying small. Again raises the question which just came up for polynomals. Matter of this paper is to specify a method for the determination of the a_i under the conditon that it is a calculus of variation problem, a condition which is given in a large number of physical and mechanical problems. The integration of the problem, in a form particularly useful for the application, is given if, under suitable choice of the ψ_i, the precision can be increased indefinitely for growing n. If we restrict ourselves initially to an independent variable and if

$$(2.)\qquad J = \int_a^b f(x, \omega, \omega', \omega'', \ldots \omega^{(\sigma)})dx$$

is the integral to be varied, then the method for the calculation of the a_i can be depicted in the following general, if also at first a little undetermined way:
At first write the expression (1.) of ω_n for ω in f; then the integral becomes a well known function $J_n(a_1, a_2, \ldots a_n)$ of the a_i, which doesn't contain x. Now determine the a_i in such a way that J_n becomes an extremum, so out of the equation system

$$(3.)\qquad \frac{\partial J_n}{\partial a_1} = 0, \frac{\partial J_n}{\partial a_2} = 0, \ldots \frac{\partial J_n}{\partial a_n} = 0$$

It is assumed here that the ψ_i are chosen in such a way, that ω_n meets the prescribed side-conditions for every value of the a_i - as far as those side-conditons don't arise out of the variation itself [1]. If the system (3.) only allows one solution $a_1 = \alpha_1^{(n)}, a_2 = \alpha_2^{(n)}, \ldots a_n = \alpha_n^{(n)}$ then

$$(4.) \qquad \omega_n(x) = \psi_0 + \alpha_1^{(n)}\psi_1 + \alpha_2^{(n)}\psi_2 + \cdots + \alpha_n^{(n)}\psi_n$$

is the best possible representation of ω, if we use the integral J itself, as the measure of precision. Wich indeed, with this determination of the a_i, won't vary a lot from its accurate minimal or maximal value $J^{(o)}$. Since for physical and mechanical problems, J is in simple connection with important variables, ie. the potential and kinetic energy (for example in the case of equilibrium it is equal to the first) it is probably justified to take this, for that operation essential measurand, as the measure fo the total error. The assumed restriciton to one variable x and one unknown function ω of course, is quite insignificant, and it turns out, as will be seen in the following, that this method for the determination of the a_i is viable, both theoretically and practically. For the boundary-value problems of mathematical physics linear diffferential equations are mostly used. J_n then is a second degree function of the a_i and the equations (3.) become linear. In all cases where the Integral J is build over a definite form - as it particularly is with equilibrum problems, the dirichlet boundary problem etc. - exists one and only one solution to the system (3.). For all functions meeting the conditions the values of such an integral possess either an (greatest) lower bound or an (least) upper bound - weather they really are achived remains unknown initally. If we now form the successive approximations $\omega_1 = \psi_0 + \alpha_1^{(1)}\psi_1; \omega_2 = \psi_0 + \alpha_1^{(2)}\psi_1 + \alpha_2^{[2]}$ etc., and the corresponding minimal values $J_1^{(1)}, J_2^{(1)}, J_3^{(1)} \ldots$, then those values form an ever decreasing or an ever increasing series of numbers, which converges towards one certain limit. It shows that the existence of this limit is sufficient proof, that the ω_n or some indefinite integrals, extended over these functions, converge uniformly in the whole given range.

[1] Such conditions, for example, are the equations at the free edges of a elastic plate, as described in Kirchhoffs laws.

This convergence even exists if the ψ_i, for example,are even functions while the sought solution ω isn't even: in this and similiar cases the ω_n converge to another limit function. But if the ψ_i have the above mentioned property, that any function ω, which satisfies the boundary and continuity conditons can be displayed approximated through an expression of the form (1.), together with a certain number of its derivatives, so the ω_n converge to the sought solution. The proof of this last sentence is essentially based on the new variation method of Mr. Hilbert. [2] The given method for successive approximations is very close to the so-called Dirichlet principle; and in every case where it is applicable the method gives a proof for the principle, because the here made particular choice of functions,which makes the to be varied integral J smaller and smaller, enables a proof of convergence. The numerical feasibility and the practical value of this method are depending mainly on the suitable choice of the ψ_i, wich of course can be made easy through the results of the experiment. In the examples I calculated (see below) they are quite satisfactory. The explanation of this necessarily somewhat vague general observation is best done with reference to some specific problems; at the same time we can determine how we have to choose the ψ_i so that the boundary and continuity conditons are met.
1^0. As first example the §§ 1 to 13 give the general solution of the previously only in very special cases treated task of calculating the deformation of an all around clamped, once flat, elastic plate of given shape under the influence of given compressive forces. This problem is equivalent to the integration of the differential equation

$$\frac{\partial^4\omega}{\partial x^4} + 2\frac{\partial^4\omega}{\partial x^2\partial y^2} + \frac{\partial^4\omega}{\partial y^4} \equiv \Delta\Delta\omega = f(x,y)$$

where f is given and $\omega = 0, \frac{\partial\omega}{\partial n} = 0$ is dictated at the boundary (n = normal vector to the boundary).

[2]D. Hilbert. "Über das Dirichletsche Prinzip" Festschrift der Kgl. Gesellschaft der Wissensch. zu Göttingen, math.-physik. Klasse, Berlin 1901, Page 17ff

2^0. § 13 and § 14 contain the solution to the dirichlet problem in its classic form: The equation $\Delta\omega = 0$ has to be fullfilled under the condition of the continuity of ω, $\frac{\partial\omega}{\partial x}$, $\frac{\partial\omega}{\partial y}$ and for given boundary values $\bar{\omega}$. The restrictions which have to be made here for the shape of the boundary curve and for the given functions $f.\bar{\omega}$ are of very general nature and quite marginal for the application.
3^0. § 15 contains the methods applicaiton for *ordinary linear differential equations with variable coefficients* which arises out of a va riation problem, where the values of the integral and possibly of some of the derivaitons are prescribed at the endpoints of the interval $a \ldots b$.
4^0. Eventually the application points to the *vibrating string*, § 16, so that the method stays numerically very useful in cases where the upper proof of convergence fails, and can be used for example for the calculation of the *Chladni plates* etc..
Within mechanics, the equations (3.) in general do not become linear, and the calculation of higher approximations gets very difficult, when applying the *hamiltonian principle* on a finite time interval. If a sufficient approximated solution is known, then the method will only be used for the calculation of the correction in this and also in other cases and we can restrict ourselves to the square members in the expansion of $J_n(a_1, a_2, \ldots, a_n)$ near the zero values of the variable. If the proper conditions for the infinite are dictated, which have to be met by the ψ_i, so that the over an infinitely large domain built integral $J and the J_n$ stay finite, then the method can also be applied. It is essential to notice, that the ψ_i can be different analytic functions in different domains respectively can be given through different expressions, if certain conditions of continuity (see for example § 2) are only fulfilled on the boundary of two such domains. Herein may lies a great relief for the application on experimental results.

2 Deformation of an at the edge clamped elastic plate under the influence of a given normal compressive force.

§ 1.

First we take care of the problem of integrating the equation

$$(5.) \qquad \Delta\Delta\omega \equiv \frac{\partial^4\omega}{\partial x^4} + 2\frac{\partial^4\omega}{\partial x^2 \partial y^2} + \frac{\partial^4\omega}{\partial y^4} = f(x, y),$$

in which, on the limit L of the given domain R the conditions have to be met

$$(6.) \qquad \bar{\omega} = 0, \frac{\partial\bar{\omega}}{\partial n} = 0$$

also the finiteness and the continuity of ω and its derivatives up to the 4. order in R and on L are required. The problem reduces itself, as one sees easily, to the minimiation of the Integral

$$(7.) \qquad J = \iint\limits_R [\frac{1}{2}(\Delta\omega)^2 - f(x, y)\omega]dS,$$

which is to be extended over the whole plate i.e. the whole domain R, under the mentioned side conditions. Up to a constant factor, J is the potential energy of the deformation[3] and f is the per unit area exerted pressure. We assume f to be finite and continuous, and that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are finite.

[3]The Kirchhoff expression for the potential energy of the plate however, contains yet another member, which vanishes identically in the case of $\bar{\omega} = 0$, $\frac{\partial\bar{\omega}}{n} = 0$, as shown by two succsessive partial integrations.

The expantion of Green's theorems on the equation (5.) is given bei Mathieu [4].
If U, V and their derivatives up to the third order, are two in the domain R, continuous functions, then

$$\iint\limits_R [U\Delta\Delta V - V\Delta\Delta U]dS = \int\limits_L [\Delta U\frac{\partial V}{\partial n} - V\frac{\partial \Delta U}{\partial n} - \Delta V\frac{\partial U}{\partial n} + U\frac{\partial \Delta V}{\partial n}]ds,$$

shall apply, with n as the outward normal to the contour L.

Now (a,b) shall be an interior point of R, and $r = \sqrt{(x-a)^2 + (y-b)^2}$; we set $V = r^2 \log r$. This function playes the same role here as $\log r$ does in the theory of potential. Hence

$$-8\pi U(a,b) = \iint\limits_R r^2 \log r \Delta\Delta U dS - \int\limits_L [r^2 \log r\frac{\partial \Delta U}{\partial n} - \frac{\partial (r^2 \log r)}{\partial n}\Delta U + 4\log r\frac{\partial U}{\partial n} - 4\frac{\partial \log r}{\partial n}U]ds.$$

So if

$$\Delta\Delta U = f(x,y); U = 0, \frac{\partial U}{\partial n} = 0 \text{ on s},$$

then

$$-8\pi U(a,b) = \iint\limits_R r^2 \log r f(x,y) dS + \int\limits_L [\Delta U\frac{\partial (r^2 \log r)}{\partial n} - \frac{\partial \Delta U}{\partial n} r^2 \log r]ds$$

shall apply.

If (a,b) is an interior point, then the line integral can be differentiated any number of times under the integral sign; the surface integral can also be differentiated three times without loosing its convergence in the point (a,b). To get derivatives of higher order, a method of Riemann[5] can be generalized:

[4] Mathieu, Journal de Lioville XIV 1869, p. 378; Théorie du potentiel Kap. III, p. 70 Paris 1890; also compare to W. Voigt, Kompendium der theor. Physik I, S. 206, Leipzig 1895.

[5] Riemann, Schwere, Elektrizität u. Magnetismus, published by Hattendorf.

It is

$$\frac{\partial}{\partial a}\iint r^2\log r f(x,y)dS=\iint\frac{\partial}{\partial a}r^2\log r f dS=-\iint\frac{\partial}{\partial x}(r^2\log r)f dS$$

$$=-\int_L r^2\log r f\cos nx ds+\iint_R r^2\log r\frac{\partial f}{\partial x}dS,$$

while $\frac{\partial f}{\partial x}$ has to be finit and f continuous. By repetition of this transformation one gets the following theorem:
If $\frac{\partial^{p+q}f}{\partial x^p\partial y^q}(p=0,1,2,...\mu;q=0,1,2,...\nu)$ are finit and continuous — $\frac{\partial^{\mu+\nu}f}{\partial x^\mu\partial y^\nu}$ only has to be finit — then you get the corresponding derivatives of U with respect to a and b; further one can get the first, second and third derivative of such a calculated derivative, through differentiation under the integral sign without transformation. All derivatives $\frac{\partial U^{p+q}}{\partial x^p\partial y^q}$ are continuous and finit, if $p\leqq\mu+3;q\leqq\nu+3;p+q\leqq\mu+\nu+3$.
One also has

$$\iint_R\Delta U\Delta V dS=\iint_R V\Delta\Delta U dS+\int_L[\Delta U\frac{\partial V}{\partial n}-\Delta V\frac{\partial U}{\partial n}]ds.$$

If one applies this on the variation of J,

$$\delta J=\iint_R[\Delta\omega\delta\Delta\omega-f\delta\omega]dS,$$

where $U=\omega, V=\delta\omega$, then also $\delta\omega=0,\frac{\partial\delta\omega}{\delta n}$ on L due to $\omega=0,\frac{\partial\omega}{\partial n}$ and the result is

$$\delta J=\iint[\Delta\Delta\omega-f]\delta\omega dS,$$

this the equation (5.).

Every solution of this equation apparently corresponds to a real minima of J. Further the values, which J can take for an arbitrary function ω which meets the continuouity and the boundary values, have a lower bound (which may will not be reached really). Because if one sets

$$\omega(a,b) = \omega_1 + \omega_2, \omega_1 = -\frac{1}{8\pi} \iint r^2 \log r f(x,y) dS,$$

$$J_0 = \iint_R [\frac{1}{2}(\delta\omega_1)^2 - f\omega_1] dS + \int_L [\omega_1 \frac{\partial \Delta\omega_1}{\partial n} - \frac{\partial \omega_1}{\partial n} \Delta\omega_1] ds,$$

then

$$\Delta\Delta\omega_2 = \ 0 \text{ and on L} : \omega_1 = -\omega_2, \frac{\partial \omega_1}{\partial n} = -\frac{\partial \omega_2}{\partial n}.$$

is obtained.

Thus the expression

$$J = \iint [\frac{(\Delta\omega_1)^2}{2} + \frac{(\Delta\omega_2)^2}{2} + \Delta\omega_1 \Delta\omega_2 - f(\omega_1 + \omega_2)] dS$$

gets the same as

$$J = J_0 + \frac{1}{2} \iint_R (\Delta\omega_2)^2 dS$$

because of

$$\iint_R \Delta\omega_1 \Delta\omega_2 dS = \iint_R \omega_2 \Delta\Delta\omega_1 dS + \int_L [\frac{\partial \omega_2}{\partial n} \Delta\omega_1 - \omega_2 \frac{\partial \Delta\omega_1}{\partial n}] ds$$

$$= \iint_R \omega_2 f dS + \int_L [-\frac{\partial \omega_1}{\partial n} \Delta\omega_1 + \omega_1 \frac{\partial \Delta\omega_1}{\partial n}] ds.$$

Thus $J \geqq J_0$, which is proof of a lower bound.
The solution of (5.) under certain conditions is uniquely determined, because the difference of two solutions ω, ω' would lead to:

$$\Delta\Delta(\omega - \omega') = 0; \omega - \omega' = 0, \frac{\partial \omega}{\partial n} - \frac{\partial \omega'}{\partial n} = 0 on L,$$

and through multiplication with $\omega - \omega'$ and through integration it leads to

$$\iint [\Delta(\omega - \omega')]^2 dS = 0.$$

It is thus $\Delta(\omega - \omega') = 0$ in R an due to $\omega - \omega' = 0$ on L, $\omega - \omega' = 0$ applies everywhere. I finish these preliminary theorems with the remark that, out of $\omega = 0$ on L follows that $\frac{\partial \omega}{\partial s} = 0$; and because also $\frac{\partial \omega}{\partial n} = 0$ it follows:

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial n} \cos(n, x) + \frac{\partial \omega}{\partial s} \cos(s, x) = 0; \frac{\partial \omega}{\partial y} = 0 \text{ on L.}$$

§ 2.

Let

$$\psi_1(x, y), \psi_2(x, y), ...\psi_n(x, y), ...$$

be an infinite series of functions which satisfy the following conditions:

$1^(0)$. They are unique finite and continuous functions of x and y, on the entire extention of the plate B and on its edge L. The same applies to their derivatives of the form

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \text{ (m = 0, 1, 2, 3; n = 0, 1, 2, 3)}$$

which play an important role hereinafter, an which we abbreviated as the main derivatives.
2^0. On L $\psi_i = 0$, $\frac{\partial \psi_i}{\partial n} = 0$ for every i.
3^0. Let ξ be an arbitrary, besides its main derivatives finit and continuous function, which is 0 everywhere besides a rectangle ρ_i which lies whole in R and its location and shape stay arbitrarily. We assume that one can choose the coefficients α_i and the index m in the expression

$$\xi_m(x, y) = \alpha_1 \psi_1 + \alpha_2 \psi_2 + ... + \alpha_m \psi_m,$$

such, that $\xi - \xi_m$ and its main derivatives stay smaller in its absolute value than an arbitrary value ϵ on R and L.

So one can specify the series $\xi_1, \xi_2, ... \xi_m, ...$ of functions, which converges uniformly to ξ in R, so do their main derivatives. (So this assumes, that the ψ_i have the properties of the polynomials, fourier series etc. to display an arbitrary function.)
4^0. A sum of the form ξ_m can only vanish identically everywhere in R, if $\alpha_1 = \alpha_2 = ... = \alpha_m = 0$. It will be shown below how, the known properties of the polynomials, fourier series etc. allow to build such functions ψ_i, with given boundary.

In order to obtain the sought successive approximation, we have to substitute

$$\omega_m = a_1\psi_1 + a_2\psi_2 + ... + a_m\psi_m$$

for ω in the integral J. If

$$(8.) \qquad J_m = \iint\limits_R [\frac{1}{2}(\Delta\omega_m)^2 - f\omega_m]dS,$$

then J_m shall be a second degree function of the a_i, which is independent of x, y

$$J_m = \sum_1^m \sum_1^m a_{pq} a_p a_q - \sum_1^m \alpha_q a_q,$$

where

$$(9.) \qquad \alpha_{pq} = \alpha_{qp} = \iint\limits_R \Delta\psi_p \Delta\psi_q dS = \iint\limits_R \Delta\Delta\psi_p \psi_q dS = \iint\limits_R \Delta\Delta\psi_q \psi_p dS,$$

$$(10.) \qquad a_p = \iint\limits_R f(x, y)\psi_p dS.$$

We choose the a_i in such a way that, $J_m(a_1, a_2, ...a_m)$ becomes a minima, in such a way that

$$(11.) \qquad \sum_{p=1}^{m} \alpha_{pq} a_p = \alpha_q. \qquad \text{(q= 1, 2, ... m)}$$

The square form

$$Q = \sum_1^m \sum_1^m \alpha_{pq} a_p a_q 0 \frac{1}{2} \iint [\Delta\omega_m]^2 dS$$

is always positive and only becomes equal to zero, if $\Delta\omega_m = 0$ in R, thus, due to $\omega_m = 0$ on L, if ω_m equals zero. Consequently according to 4^0, that $a_1, a_2, ...a_m$ have to vanish, if $Q = 0$.

Due to J_m having a lower bound J_0 (as we have shown), J_m reaches a minima for a uniquely defined value system of the a_i, meaning that the equations (11) are only solvable in one way, i.e. their determinants are different from zero.
Besides it should be mentioned, that also in cases where $J_m(a_1, a_2, ...a_m)$ has a more general analytic form, the minima, if existing, is really reached according to a known theorem, so that for a finit number of parameters there won't be any problems, like there are within the "Dirichlet's principle", where the number of parameters can reach the infinit.
Let $d'\omega$ be the total differential of ω with respekt to the a_i and

$$\xi_m = \sum_1^m A_p \psi_p,$$

where the A_p are arbitrarily; then one can set $da_p = \epsilon A_p$ (ϵ = infinitesimal value), so that

$$d'\omega_m = \epsilon\xi_m$$

is an expression of the same form of ω_m. Out of

$$(12.) \qquad d'J = \sum_1^m \frac{\partial J}{\partial a_p} da_p = 0 = \iint [\Delta\omega_m \Delta d'\omega_m - f d'\omega_m] dS,$$

we get

$$(13.) \qquad 0 = \iint [\Delta\omega_m \Delta\xi_m - f\xi_m] dS$$

due to the boundary conditions fullfilled by ψ_i. Omitting the faktor ϵ follows

$$(14.) \qquad 0 = \iint [\Delta\Delta\omega_m - f]\xi_m dS.$$

Particularly setting $\xi_m = \omega_m$, so that (13.) gives us

$$\iint_R [(\Delta\omega_m)^2 - f\omega_m] dS = 0.$$

in such a way, that using (7.)

$$(15.) \qquad \min \mathcal{J}_m = J_m^{(0)} = -\frac{1}{2} \iint_R [\Delta\omega_m]^2 dS.$$

In the same way one can find a new approximation, corresponding to the index $m+n > m$, with a new value system a_i' of the a_i. If

$$a_p' - a_p = \epsilon_p (p = 1, 2, ...m); a_p' = \epsilon_p \; for \; p = m+1...m+n,$$

$$\omega_{m+n} - \omega_m = \phi_{mn} = \sum_1^{m+n} \epsilon_p \psi_p; \quad \xi_{m+n} = \sum_1^{m+n} A_p' \psi_p,$$

then following shall apply for the minimum $J_{m+n}^{(0)}$

$$(16.) \qquad J_{m+n}^{(0)} = \iint \{\frac{1}{2}[\Delta\omega_m + \Delta\phi_{mn}]^2 - f[\omega_m + \phi_{mn}]\} dS$$

$$= J_m^{(0)} + \iint \{\Delta\omega_m \Delta\phi_{mn} + \frac{1}{2}(\Delta\phi_{mn})^2 - f\phi_{mn}\} dS$$

and

$$0 = \iint \{[\Delta\omega_m + \Delta\phi_{mn}]\Delta\xi_{m+n} - f\xi_{m+n}\} dS.$$

If one sets $A'_p = \epsilon_p$, i.e. $\xi_{m+n} = \psi_{m+n}$, then one gets a new equation which, if used in (16.), gives us

$$(17.) \qquad J^{(0)}_{m+n} = J^{(0)}_m - \frac{1}{2} \iint\limits_R (\Delta\psi_{mn})^2 dS.$$

So the values $J^{(0)}_1, J^{(0)}_2, ...$ form a a never-increasing infinite series of numbers, as was to be expected a priori. Due to the series staying bigger then J_0, it has a boundary $J^{(0)} \geqq J_0$, and one can define a number M in such a way, that if m $>$ M and η is an even smaller given number, then for every n

$$(18.) \qquad 2 \left| J^{(0)}_{m+n} - J^{(0)}_m \right| = \iint (\Delta\phi_{mn})^2 dS < \eta.$$

From this inequality we can infer the convergence of our successive approximations.

§ 3.

If

$$\phi = \frac{\phi_{mn}}{\sqrt{\eta}},$$

then due to (18.)

$$(19.) \qquad \iint\limits_R (\Delta\phi)^2 dS < 1.$$

It is identical

$$(20.) \qquad \phi(a,b) = -\frac{1}{2\pi} \iint\limits_R \log r \Delta\phi dS \qquad\qquad (r^2 = (x-a)^2 + (y-b)^2)$$

for every interior point, because the additional term $\int\limits_L [\log r \frac{\partial\phi}{\partial n} - \phi \frac{\partial \log r}{\partial n}] ds$, which is added in the general, equals zero due to $\phi = 0$, $\frac{\partial\phi}{\partial n} = 0$ on L. D shall be the domain in which $|\log r| \leqq |\Delta\phi|$, and D' the one in which $|\Delta\phi| < \log r$ — the last one contains the point a b — then

$$2\pi\,|\phi| < \iint\limits_R |\log r|\,|\Delta\phi|\,dS < \iint\limits_D (\Delta\phi)^2 dS + \iint\limits_{D'} (\log r)^2 dS$$

$$< \iint\limits_R (\Delta\phi)^2 dS + \iint\limits_R \log^2 r dS.$$

The last integral is convergent, as can be seen by introducing polar coordinates, with $r\log^2 r$ converging to zero for $r = 0$; it depends on (a, b); if K is its biggest value in R, so eventually

$$|\phi| < \frac{1+K}{2\pi}$$

and thus

$$|\phi_{mn}| = |\omega_{m+n} - \omega_m| < \frac{1+K}{2\pi}\sqrt{\eta}.$$

The value $\frac{1+K}{2\pi}\sqrt{\eta}$ is independent of x, y and n; it can bise made smaller then any given number ϵ, due to the convergence of $J_m^{(0)}$ one can always choose a M so that,

$$\eta \leqq \frac{4\pi^2}{(1+K)^2}\epsilon^2.$$

Then $|\omega_{m+n} - \omega_m| < \epsilon$ applies for every $m > M$ and every n, x and y. This gives us the theorem:
The series of functions $\omega_1, \omega_2, \omega_3, ...$ converges uniformly against a function $\omega(x, y)$, in the whole domain R, the function is thereby continuous in R and vanishes on its boundary L.
This indeed is proof of the convergence of our successive approximations; but because we have not made use of the prerequisite (3.), the function ω can differ a lot from the sought solution, when choosing the ψ_i very bad. If one for example has chosen an even function for the ψ_i while the solution is an odd function, then ω, of course, can not be the solution. It is noteworthy, that the mere fact that we define the infinite number of constants that we have, according to the given schema, in its own right, under all circumstances, forces the convergence.

That, what we are going to show, the right boundary function ω comes out, if the ω_m, in analogy to the polynomials, the fourier series etc., can depict almost arbitrary functions, is to be expected a priori. This condition is not a necessary one, particularly if it is only about reaching a certain grade of precision: like with semi-convergent series an optimum of the approximation can be reached, for a given m. For numerical applications it is not usually to ensure that the made ansatz

$$a_1\psi_1 + a_2\psi_2 + ...a_n\psi_n$$

can interpolate any function with arbitrary precision, rather it should depict a function of the sought solutions form, with the prescribed precision.
From the fact that the integral (18.) gets infinitely small with increasing m, one, of course, can not derive that the integrand $(\Delta\phi_{mn})^2 = (\Delta\omega_{m+n} - \Delta\omega_m)^2$ also has this property, and therby, that the derivatives of first and second order do also converge.

However, with polynomials etc., in cases appearing in the application,one will not likely be brought to functions which stay finit in single points, but become infinitesimal in the biggest part of the domain R, like it is necessary for the integral to become infinitesmal too. Rather the integrand itself will stay small, and therby the first and second derivative will be depicted, arbitrarily approximated, by $\frac{\partial\omega_m}{\partial x}, ...,$. But if each function, sufficient to the boundary and continuity conditions, together with its first and second derivatives, can be represented, in R, within any precision, by an expression of the form (21.) and its derivatives, then the lower bound of the $J_m^{(0)}$ can not differ from the lower bound of J for arbitrary functions, sufficient to the conditions, and the limit function ω is the sought solution in that case.

In the following, we will provide the strict proof, that $\lim \omega_m$ is the sought solution, by using the not yet used condition 3^0 of page 10. This condition 3^0 however, requires slightly more than generally necessary for the numerical application.

It can be initally still be shown in a general way that, for α, β an interior point of R, the limits do also exist

$$\lim_{m=\infty} \frac{\partial}{\partial y} \int\limits_{\alpha}^{x} \omega_m dx = \frac{\partial}{\partial y} \int\limits_{\alpha}^{x} \omega dx; \quad \lim_{m=\infty} \frac{\partial}{\partial x} \int\limits_{\beta}^{y} \omega_m dy = \frac{\partial}{\partial x} \int\limits_{\beta}^{y} \omega dy.$$

This proof again is build on,

$$\int\limits_{\alpha}^{x} \frac{\partial \phi(x,y)}{\partial y} dx; \quad \int\limits_{\beta}^{y} \frac{\partial \phi}{\partial x} dy,$$

in which the integration paths $\alpha...x$, $\beta...y$ lie in R, in consequence of (19.), staying smaller than given, only on the shape of the plate depending numbers. Because one can first integrate (20.) with respect to a and then differentiate with respect to b, under the Integral sign; now

$$\frac{\partial}{\partial b} \int\limits_{\alpha}^{a} \log r \; da = \arctan \frac{a-x}{b-y} - \arctan \frac{a-x}{b-y},$$

even if the integration path leads through the point $x = a, y = b$, if only a and α are different. Due to the right side beeing always $<2\pi$, it follows that

$$\left| \frac{\partial}{\partial b} \int\limits_{\alpha}^{a} \phi(a,b) da \right| < \iint\limits_{R} |\Delta \phi| \, dS.$$

Now one seperates R again into an area D, in which $|\Delta\phi| \leqq 1$ and D' in which $|\Delta\phi| > 1$; it follows

$$\iint\limits_{D} |\Delta \phi| \, dS < \iint\limits_{D} dS < \text{Oberfläche von R},$$

$$\iint\limits_{D'} |\Delta \phi| \, dS < \iint\limits_{D'} |\Delta \phi|^2 \, dS < 1,$$

thus

$$\left| \frac{\partial}{\partial b} \int\limits_{\alpha}^{a} \phi(a,b) da \right| < \ 1 + \text{Oberfläche von R},$$

whereby, as above, the convergence theorem for

$$\frac{\partial}{\partial y} \int\limits_{\alpha}^{x} \omega_m ds; \quad \frac{\partial}{\partial x} \int\limits_{\beta}^{y} \omega_m dy$$

is shown. That the corresponding limit functions do not differ from

$$(22.) \qquad \frac{\partial}{\partial y} \int\limits_{\alpha}^{x} \omega dx; \quad \frac{\partial}{\partial x} \int\limits_{\beta}^{y} \omega dy$$

and thus, even if not ω, but that the integrals over ω have first derivatives, is immediately obvious, because due to the uniformity of convergence for each rectangle in R

$$\int\limits_{\beta}^{y} dy \lim_{m=\infty} \int\limits_{\alpha}^{x} \frac{\partial \omega_m}{\partial y} dx = \lim_{m=\infty} \int\limits_{\alpha}^{x} [\omega_m(x,y) - \omega_m(x,\beta)] dx$$

$$= \int\limits_{a}^{x} [\omega(x,y) - \omega(x,\beta)] dx.$$

Since the differential quotient of y to the left side

$$\lim_{m=\infty} \int\limits_{\alpha}^{x} \frac{\partial \omega_m}{\partial y} dx$$

does exist, it implies that also $\int\limits_{a}^{x} \omega(x,y) dx$ is differentiable for y and equals the sought limit. However the sequence of differentiation and integration in (22.) may must not be reversed. It also follows generally from

$$\left| \int\limits_{L} \frac{\partial \phi}{\partial n} ds \right| = \left| \iint\limits_{R} \Delta \phi dS \right| \leqq \iint\limits_{R} |\Delta \phi| \, dS \leqq \text{given number}$$

by application to a region lying within R, which is bounded by a curve segment l and two parallels to the axes, that

$$\int\limits_{l} \frac{\partial \phi}{\partial n} ds < \text{ given number}$$

due to the same has already been shown for the parallels to the axes. Thereby l can also be part of L. The limit

$$\lim_{m=\infty} \int\limits_{l} \frac{\partial \omega_m}{\partial n} ds$$

does exist and it equals zero on L, due to $\frac{\partial \omega_m}{\partial n} = 0$ for all m.